SECTION DE MATHÉMATIQUES EPFL

MATH-111(en)
Linear Algebra

FALL 2025
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Graded Exercise 1

2 October 2025

Rules:

- Turn in only these stapled pages (i.e. keep your solution concise but complete!)
- In the top right corner of your piece of paper, please write your SCIPER (or any other 6 digit number that you will recognize when we're handing the paper back to you).
- Handwriting must be legible (else your solution won't be graded).
- Hand in by end of lecture or email to Nicolas Pafumi + Souhail Ed-Dlimi.

Problem 1

Consider the equation $Ax = b$ where

$$A = \begin{pmatrix} 10 & 30 + c_1 \\ -1 & -3 \end{pmatrix} \text{ and } b = \begin{pmatrix} c_2 \\ 0 \end{pmatrix}$$

For every pair of constants $c_1, c_2 \in \mathbb{R}$, find the solution space for the equation $Ax = b$.

Problem 2

Assume that $\{v, w\} \subset \mathbb{R}^n$ are linearly independent.

a) Prove that $\{v + w, 3w\}$ are linearly independent.

b) Can there exist a vector $u \in \mathbb{R}^n$ such that $v, w \in \text{Span}\{u\}$? Justify your answer.

BE AWARE : "Grade" just means that we provide you with feedback / corrections on your submission. It does not mean that you will receive a numeric grade. This is an exercise and does not count towards the final grade of this course. Participation is not mandatory.

2) b) No! Otherwise $\exists\ \alpha, \beta \in \mathbb{R}$ s.t. $V = \alpha u$ and $W = \beta u$

↳ if $\alpha = \beta = 0$ then $V = W = 0$ so they can't be independent $(1 \cdot W + 0 \cdot V = 0)$

↳ if $(\alpha, \beta) \neq (0,0)$ then $\beta \cdot V - \alpha W = \beta \alpha u - \alpha \beta u = 0$ so $\{V, W\}$ can't be independent

1) Gaussian elimination

$$\left(\begin{array}{cc|c} 10 & 30+c_1 & c_2 \\ \boxed{-1} & -3 & 0 \end{array}\right) \xrightarrow{R2\times(-1)} \left(\begin{array}{cc|c} 10 & 30+c_1 & c_2 \\ \boxed{1} & 3 & 0 \end{array}\right) \xrightarrow{\text{swap } R1 \leftrightarrow R2}$$

$$\left(\begin{array}{cc|c} \boxed{1} & 3 & 0 \\ 10 & 30+c_1 & c_2 \end{array}\right) \xrightarrow[\text{from } R2]{\text{subtract } 10\times R1} \left(\begin{array}{cc|c} \boxed{1} & 3 & 0 \\ 0 & c_1 & c_2 \end{array}\right)$$

Discussion: if $c_1=0$, matrix is already in reduced echelon form

→ if $c_2\neq 0$ we have a row $(0\ 0 \mid \neq 0)$ so solution space is $\emptyset$ = empty set

→ if $c_2=0$ we have a row $(0\ 0 \mid 0)$ so we can just ignore it and solve

$$(\boxed{1}\ \ 3 \mid 0) \iff x_1+3x_2=0$$

$$\iff x_2=s \ \& \ x_1=-3s, \ \forall s\in\mathbb{R}$$

So solution space is $\left\{\begin{pmatrix}-3s\\ s\end{pmatrix}, s\in\mathbb{R}\right\}$

if $c_1\neq 0$, divide second row by c_1

$$\left(\begin{array}{cc|c} \boxed{1} & 3 & 0 \\ 0 & \boxed{1} & \frac{c_2}{c_1} \end{array}\right) \xrightarrow[\text{from } R1]{\text{subtract } 3R2} \left(\begin{array}{cc|c} \boxed{1} & 0 & -\frac{3c_2}{c_1} \\ 0 & \boxed{1} & \frac{c_2}{c_1} \end{array}\right)$$

So solution space is $\left\{\begin{pmatrix}-3c_2/c_1\\ c_2/c_1\end{pmatrix}\right\}$ = one-element set

2) a) assume for purpose of contradiction that $\{v+w, 3w\}$ are dependent $\Rightarrow \exists\, \alpha,\beta\in\mathbb{R}$ not both 0 such that $\alpha(v+w)+\beta\cdot 3w=0$

$\Updownarrow$

$\{v,w\}$ are dependent since $\alpha, \alpha+3\beta$ can't both be 0 $\Leftarrow\ \alpha v+w(\alpha+3\beta)=0$

Contradiction! So $\{v+w, 3w\}$ are linearly independent

~~b) No! Otherwise there would exist $\alpha,\beta\in\mathbb{R}$ such that $v=\alpha u$, $w=\beta u$ and we would have $\beta v-\alpha w=0$ (moreover, if $\alpha,\beta=0$ then $v=w=0$ so they can't be independent)~~